Tutorial for

Wesley B. Loewer's

Plotting Programs

version 2.51

Tutorial Updated:

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Introduction

WL-Plot was designed to help students understand graphs a little better by allowing them to quickly plot a graph to see what the function looks like. Often, students get bogged down in button pushing and point-by-point plotting and fail to grasp the real significance behind the graph. The goal of this program is to eliminate the busy work of plotting, freeing the student to proceed to higher level thinking.

WL-Plot goes well beyond just graphing functions. Some of the illustrations that I have been able to show to my class would have simply been impossible to demonstrate with out WL-Plot. Some of its features demonstrate mathematical concepts that perhaps are not a part of your course's normal curriculum. I have found that these topics are usually the ones that spark students' interest. Therefore, I hope that you will take the time to learn all of WL-Plot's features.

What follows is a tutorial to help students and teachers become familiar with the many features of WL-Plot and how it can be used in the classroom. It is important to realize that WL-Plot itself is a tool much like a reference book or calculator. WL-Plot is not a tutorial in that it does not contain exercises, ask the student questions, nor check their progress. It is, however, a very powerful tool that can be used in a variety of ways to demonstrate various mathematical concepts.

Enjoy,

Wesley B. Loewer

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Getting Started

Before you use WL-Plot for the first time, you must copy all the files to a disk or subdirectory on your hard drive. Move to the drive or directory where the WL-Plot program is located and type "setup" to set things up for your system. To start WL-Plot, simply type "wlplot" without the quotes. A menu will be presented with all of the different graphing modules available. Press the up or down arrow keys till you reach the desired selection, or if you have a mouse move the mouse cursor to the desired selection and click the left mouse button. If at any time in the program you are not certain what a particular option does, move to that option and press the <F1> key for help.

Functions (and a little history)

WL-Plot got its start in the 1989-90 school year when I was teaching Algebra II for the very first time. I owned a graphing calculator that I had purchased in 1987, but they had not yet become as popular and affordable as they are now, so very few students had access to them. I was excited when I found that the department had a computerized graphing program that I could use for demonstrations in front of the class. Unfortunately, it was so cumbersome to use and its display of the graphs was so confusing that the students would literally groan when I turned the computer on to use it. Being a computer programmer myself, I thought, "Surely I could do better than this!" Hence, WL-Plot was born as simply a function plotter. As the years went by, WL-Plot has been enhanced by more and more features, but the function plotting module still remains the heart of the program.

Much has been written in recent years on the advantages of graphing programs and graphing calculators. If you have access to a classroom set of graphing calculators, much of what is done in the function plotting module of WL-Plot can be done on the calculator. The speed and high resolution of a computer can make WL-Plot's function plotter worthwhile however.

Entering a function

When you first start the functions module, the cursor will be blinking in the upper left hand corner of the screen. It is waiting for you to enter a function. Let's start with a simple equation: $k=1/2*m*v^2$. Simply enter the right half of the equation as "1/2*mass*vel^2" without the quotes. (You could have also entered "1/2*mass*vel^2" using <Ctrl-2> for the "^{2}" character as is indicated in the <F1> help screen.) Notice that you can enter actual words as your variables. At this point the computer assumes that the first variable in the equation, mass, is to be the independent variable for the horizontal axis. You will be shown how to change this a little later. To indicate what the value of the vel variable should be, move over to the section labeled

"<F6> Constants" by pressing the <F6> key and arrowing down to the word "vel" or clicking the left mouse button on the word "vel". Now type in "2.4" in for the velocity value.

We are now ready to graph our first function. At the lower left of the screen you will see a section of options that starts with "<F8> Plot". You can press the <F8> key to select that section and then press the <Enter> key to select Plot. Since this option is so often used, there is a short-cut to select Plot. Simply press the <Ctrl-Enter> key (that is, while holding down the <Ctrl> key, press the <Enter> key), or press the <Right-Mouse> button. Go ahead and select Plot using one of the above methods. You should see a graph that looks like the one on the right. To get back the previous menu, press the <Enter> key or the <Right-Mouse> button.

In order to change the value of the velocity, move back to the word "vel" if you are not already there by using the mouse or pressing <F6>. Now simply type in a new value of "0.6" and plot it again. One of WL-Plot's conveniences is the ease at which the values can be changed and re-graphed. You could have originally typed in "1/2*mass*2.4^2" instead of using constants, but then changing it to "1/2*mass*0.6^2" would have required editing the function instead of just entering in a new value for the velocity.

The resulting graphs are lines which extend from the left side of the screen to the right. This is expected since the mass was not raised to a power. However, since mass cannot be negative, it does not make sense to graph values less than zero. Normally, WL-Plot bases the domain of the independent variable on the coordinates of the screen. However, this feature can be turned off. Use the mouse or press the <F7> key and arrow down to the "Domain based on..." option. To turn it off, click the <Left-Mouse> button or press <Enter>. Then arrow up to "mass min" and enter the value 0. Now plot the graph to see that the graph starts at the origin. To turn the automatic domain selection back on, move back to the "Domain based on..." option and press <Enter> again.

Suppose now you would like to make "vel" the independent variable. Use the mouse or press the <F6> key and arrow keys to get back to the word "vel." Instead of entering a number for vel, just type in the letter "i" and then move up to the word "mass" and enter a value for it. Try setting mass to 0.25 and plotting it. You should get a parabola as shown.

Instead of using the words "mass" and "vel" we could have used just "m" and "v" and entered the equation as "1/2*m*v^2". Implied multiplication is allowed by entering "m v^2" but you could not use "mv^2" as the computer would interpret this as a single word variable "mv" that is being squared. If you

want a two word variable, connect the two words with the underscore character such as "my_height" as the use of "my height" without the connector would be interpreted as "my" times "height".

Be careful in your use of implied multiplication. It makes the equation easier to type, but can lead to confusion. For instance, "1/2 m v^2" will be interpreted as "1/(2*m*v^2)". To get the desired results, you would have to use "(1/2)m v^2" or ".5m v^2" or "m v^2/2". Remember, on most graphing calculators and graphing programs, implied multiplication is higher in the order of operations than explicit multiplication and division. If in doubt, use parentheses or use the explicit multiplication symbol, '*'.

<u>Exploring a family of functions</u>
Often you want to be able to compare how a function's graph changes as certain constants change. For example, a commonly used equation is y=ax^n, which would be entered into the computer as "a*x^n". To enter this equation, get back the section labeled "<F2> Function:" by pressing the <F2> key or using the mouse. After entering "a*x^n", press the <F6> key to get to the Constants section. You will notice that the new variables "a", "x", and "n" have been added to the list of variables. Since the variables "mass" and "vel" are no longer needed they can be ignored, or if you would like to tidy things up a bit, arrow up to the option "Purge Unused Constants" and press the <Enter> key and they will be removed.

Arrow down to the "a" constant and let's start with setting "a" to 1. Now press <Enter> and the cursor moves down to "x". Since we want "x" to be the independent variable, type in the letter "i" and press <Enter>. Next, we'll give "n" the value 2 to look at a family of parabolas. Finally press <Ctrl-Enter> to graph the function. You should have something that looks like the graph on the right.

Overlaying Graphs

In order to compare this graph with one in which a=2, we need to turn on the Overlay Graph option. This option is under the section entitled "<F4> Plotting Style". Use the mouse or press the <F4> key and press the down arrow key to get to the Overlay Graph option and then press the <Enter> key. The same thing can be accomplished by clicking the left mouse button on that option. A mark should appear next to the option indicating that you have turned it on. Now go back and change the value of "a" from 1 to 2. When you press <Ctrl-Enter> to graph the function, you will notice that the new function is simply plotted over the previous one. Now change the value of "a" to 0.5 and plot it again. There is no limit to how many graphs can be overlaid on top of each other. Don't forget to turn off the Overlay Graph option when you are finished. In the same manner, the value of "a" could have been held constant and the value of "n" could have been changed to compare squared terms to cubics, etc...

Simultaneous Equations

One of the things that function plotters are useful for is determining where two functions intersect. In order to graph two equations simultaneously, move the section labeled "<F3> Plot Type:" by pressing the <F3> key. Arrow to the option called "Cartesian" and press the <Enter> key. You should notice that it changes from "(1) Cartesian" to "(2) Cartesian" indicating that two graphs Cartesian equations can be entered. Press <F2> to get back to the Functions sections and type in the equation "x^3" then press <Enter> and then type in the equation "2^x". Since "x" is still the independent variable, no changes are necessary in the Constants section, although you could Purge Unused Constants if you wanted. Make sure the Overlay Graph option is turned off and then plot the two functions. Don't forget, <Ctrl-Enter> to plot, <Enter> or <Esc> to get back. As always, if you do forget, just hit the <F1> help key. The two functions should intersect in the first quadrant.

Grid Lines and (X,Y) Coordinates

One thing that would be useful here is if the graph had grid lines. After returning to the menu, press the <F4> key to get back to the Plotting Style section and then press the down arrow key till you reach the Grid option.

Press the <Enter> key to turn the Grid option on and plot the graph again. You can now clearly see that they intersect when x is between 1 and 2 while y is between 2 and 3. To determine a more exact values of (x,y), move the arrow on the graph to the point of intersection. This is done with either the arrow keys or the mouse. To move the arrow faster, hold the <Shift> key down before pressing the arrow keys. When you have the arrow as close as possible to the point of intersection, press the space bar and you will see the (x,y) coordinates given in the lower left hand corner of the screen to be about (1.36,2.52).

Zooming In
In order to see this graph more accurately, we can zoom-in. We will make a box around the area in which we want to zoom. If you have a mouse, move the mouse pointer to one corner of the region in which you would like to zoom, such as the coordinates (1,3). Then, while holding the <Left-Mouse> button down, drag the mouse to the opposite corner of the zoom region, such as the coordinates (2,1). You will notice a box is displayed, showing the location of the zoom box. If you mess up, just repeat the process. If you change your mind, just click the <Left-Mouse> button without dragging it. Once you get the box where you would like it, press the <Right-Mouse> button or press the <Enter> key and the zoomed-in functions will be plotted.

If you do not have a mouse, use the arrow keys to move the arrow up to about y=3 and press the "t" key to indicate the top of the box. Then move the arrow down to about y=2 and hit "b" to mark the bottom of the box. Next move over so that the arrow is about x=1 and hit the "l" key to mark the left edge of the box. Finally, move to the right till about x=2 and hit the "r" key to mark the right side. If you forget which key does what, just hit the <F1> key for help. Once you have defined your zoom box, just hit the <Enter> key.

Now you will see the point of intersection more accurately. Again move the arrow to the point of intersection and press the space bar to read the (x,y) coordinates. You may continue zooming in for more and more accuracy. If you wish to cancel the zoom-box and return to the menu, just press the <Esc> key.

Zooming Out:
If you ever want to zoom out, hit <Ctrl-Enter> instead of just <Enter>, or <Ctrl-Right-Mouse> instead of <Right-Mouse>, after you have defined the zoom box in which case the whole screen will be squeezed into the zoom box.

Restoring Zoom Box

After you are finished zooming in, you may want to restore the graphing area back to its original values. To do this, press <Esc> to get back to the menu and then move to the "<F5> Plotting Area:" section. You will notice that the min/max value for x and y can be entered manually instead of zooming in with the arrow keys on the graph. To change a value, press the down arrow till you get to the one you want. The first option is "Reset Plotting Area". Selecting this option will restore the plotting region to its original value.

Changing the Number of Steps
Under the Plotting Style section, there is an option called Steps. This value determines how many time the function is sampled in determining the shape of the graph. The larger the value, the more accurate the graph, but the longer it will take. The default value is usually sufficient, but you may find it necessary to increase it.

Composite Functions
Suppose you have two functions: f(x)=3x and g(x)=sin(x) and you would like to graph f(g(x)). This can be accomplished with WL-Plot by entering the equation as "sin(x)->g 3g". Technically, what this does is save the results of sin(x) in g and then multiplies 3 times g. However, this has the exact same effect as f(g(x)). Similarly, the composite function g(f(x)) would be entered as "3x->f sin(f)".

Conditional Functions
Conditional functions such as y=2x if x>2 or else y=x^2 can be entered as "if(x>2, 2x, x^2)". The general syntax for the "if" function is "if(b,t,f)" where b is some condition which is either true or false. If "b" is true, then "t" is evaluated. If "b" is false, then "f" is evaluated.

<u>Polar Graphs</u>
Besides Cartesian graphs, WL-Plot can also show graphs in polar coordinates. Move to the "<F3> Plot Type:" section and select the Polar option. Now move up to the "<F2> Function:" section and enter the two equations "sin 2Θ" and "1+sin Θ". To get the "Θ" character, use <Ctrl-T> (hold the <Ctrl> and hit the "t" key). Since Θ is now the independent variable, move to the Constants sections and enter an "i" for Θ or just select Purge Unused Constants. Now press <Ctrl-Enter> to graph the two functions. To find a point of intersection, move the arrow to the desired point and press the space bar. The (x,y) coordinates will show up in the lower left hand corner. Although (x,y) may be useful on some occasions, you probably would prefer the (r,Θ) polar coordinates. To change back and forth between Cartesian and Polar coordinates, press the "p" key (for Polar) and the (radius,angle) coordinates will be shown. To change the angular units between degrees and radians, press the "d" key (for Degrees).

By default, the functions are plotted from Θ=0 to 2π. This can be changed by moving to the "<F7> Domain:" section and entering new values for the min and max. You can enter multiples of π by entering something like "5π", using <Ctrl-P> for the "π" character.

<u>Parametric Graphs</u>
WL-Plot can also handle Parametric Graphs. Simply move the Plot Type section and select Parametric. If you then plot x(t)=5sin(2t) and y(t)=4sin(2.5t) and make "t" the independent variable, they will be plotted with t being the parametric variable. Change the domain to go from 0 to 4π and the graph on the right should be produced. All zooming, coordinates, and domain changes are handled the same as before. Especially with complicated Parametric Graphs, you may need to increase the value of the Steps option under the Plotting Style section.

<u>Saving your work</u>
If you want to save a particular equation along with the constant values and plotting style, etc... select Save by pressing the <F8> key, arrowing over to "Save" and pressing <Enter>. You will be prompted for a file name which can be up to 8 letters with no spaces, periods, question marks, or asterix. To retrieve it later, just select the "Retrieve" option and enter the same file name. To save the actual picture of the graph, select the "Save-PCX" option. PCX files can be used in other programs such as word processors and paint programs. This is very useful if you want to use a graph in a handout or on a test.

Conic Sections

A significant topic in Algebra II and Pre-Calculus courses is the subject of quadratic equations and the conic sections that are associated with them. While the function plotting module described in the previous sections can be made to plot conic sections, it involves cumbersome manual manipulation of the equations. The fact that usually one value of x has more than one y, multiple functions must be plotted in order to form the entire shape. The Conic Sections module makes this process much less painful.

When you select the Conic Sections module, a new menu is presented. One the left side of the screen a list is shown of all the shapes that can be made with the general quadratic equation. These include not only the four conic section shapes (circle, ellipse, parabola, and ellipse), but the degenerate shapes as well (intersecting lines, parallel lines, single line, and a single point).

Changing the Standard Equation Constants

Let's start with the simplest shape, the circle. Since the program starts up with the circle type already selected, you should see the equation for a circle on the right side of the screen with the constants h=0, k=0, and r=1 listed below. The result is the simple circle formula $x^2+y^2=1$, which is a circle of radius 1 centered about the origin. Further down the screen on the right side you will see the general equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ with the constants A=1, B=0, C=1, D=0, E=0, F=-1, which reduces to the equation $x^2+y^2-1=0$ which is the same as $x^2+y^2=1$, the same circle formula as before. Now move to the Standard Equation Constants section by pressing the <F6> key and change the values to h=1, k=2, r=3. As you enter the values you will notice the values of A through F in the General Equation Constants section automatically change to reflect the changes that you make in the Standard Equation Constants.

To actually see the graph, press the <Ctrl-Enter> key just like you did in the function plotting module. All the zooming in and out is the same as well. WL-Plot can also show the focal points, asymptotes, directrix, etc... To turn this option on, move to the Plotting Style section and select the Focal Points and Asymptotes option.

As with the function plotter, graphs can be overlaid on top of each other by using the Overlay graph option.

Rotating the Graph

You can also enter an angle of rotation which tells the grapher to rotate the whole graph about the origin a specified number of degrees. Alternately, you can specify the tangent of the angle. If you specify an angle of rotation,

you will notice that the Bxy term of the general is no longer zero. As an example, try the following: select the Ellipse from the selection of Conic Type and set h=1, k=2, a=3, b=4, Θ=25. When you plot it, you will see the ellipse along with its focal points and surrounding box all rotated about the origin 25°.

Changing the General Equation Constants
You can also specify the values of the general equation constants. Move to the General Equation Constants section by pressing the <F7> key and try entering the following: A=1, B=-2, C=3, D=-4, E=5, F=-6. As you enter the values, you will notice that the constants in the Standard Equation section will automatically change. In fact, depending on what values you enter, it can change from one type to another such as from an ellipse to a hyperbola or from a horizontal parabola to a vertical one. If you enter constant values such that the equation has no real solution, then no conic shape is selected at all.

Derivatives and Integrals

Often in Calculus and Physics, it is desirable to view not just the graph of a function, but its derivatives and integral as well. With this module of WL-Plot, you can view the function, its derivative, its second derivative, and its indefinite integral by itself or in any combination. The graphs actually shown are determined by the selections made under the Plotting Style section with the Show Indefinite Integral, Show Function, etc... Let's start with the commonly used free-fall formula $h=h0+v0*t+0.5*g*t^2$ (use <Ctrl-2> to get the ^{2} symbol). After entering this formula in the Function section, move over to the Constants section and set h0=2, v0=5, t=i (for independent), and g=-9.8. For a better plotting region, move over to the Plotting Area section by hitting the <F5> key. Set x min=-1, x max=2, f min=-2, f max=4, f' min=-12, f' max=12, f" min=-11, and f"=2. Now show the graph using the <Ctrl-Enter> key. The function is graphed above and the derivative below. Press the <Esc> key to get back to the menu and turn on the Show 2nd Derivative option under the Plotting Style section and show the graph again. This time the second derivative is shown, a horizontal line at -9.8.

Turning on the Show Indefinite Integral option requires addition instructions from the user. Since this is an indefinite integral, the user must determine what the integral constant should be. Under the section "<F7> Domain & Integral Constant" the user may indicate the (x,y) point through which the integral is to pass. If possible, WL-Plot will try to force the graph of the integral to pass through this point. Alternately, the use may turn on the "Integral Constant based on average" option which tries to center the graph about the x-axis.

Relation Plotter

I had a very interesting incident occur in class several years ago while studying conic sections. We were having a class discussion on why ellipses were shaped the way they were based on the definition, "the set of points such that for each point the *sum* of its distances, d_1+d_2, from two fixed points F_1 and F_2, is a constant." One particularly inquisitive student asked the question, "What would the shape look like it you defined a shape to be the set of points such that for each point the *product* of its distances, $d_1 \cdot d_2$, from two fixed points F_1 and F_2, is a constant?" Well, I hemmed and hawed a bit and made sort of a guess and told them I'd have to think about that one a bit more before I ventured an answer. Later in the day, I tried to work it out and solve for y, but you end up with a higher order polynomial to solve, which I admit is not a favorite task of mine. That night I went home and wrote a program to graph out the solution. The solution, by the way, was at first glance surprising, but later seemed so obvious. Yet without the computer it would have been very difficult to figured out.

With a graphics printout in hand, I very proudly presented it to the class the following day. After the students "ahh'ed" a bit, another student asked, "What if you had three fixed points and the sum of the distances, $d_1+d_2+d_3$, was a constant?" I realized at this point that what we needed was a program that would accept any equation and graph its shape. Hence, the Relation Plotter was born.

As the name implies, this module will graph any relation, regardless of whether or not it is a function. To demonstrate how it does this, let's start with an equation whose graph has a shape with which we are already familiar, the ellipse. The definition of an ellipse states that it is the set of points whose sum of the distances from two fixed points is a constant. If we let the two fixed points be (x1,y1) and (x2,y2), the an equation describing this situation would be $\sqrt{((x-x1)^2+(y-y1)^2)} + \sqrt{((x-x2)^2+(y-y2)^2)} = c$. Since the Relation Plotter requires that all equations be solved for 0, we'll just move the "c" to the other side and enter the following equation: "$\sqrt{((x-x1)^2+(y-y1)^2)} + \sqrt{((x-x2)^2+(y-y2)^2)} - c$" using <Ctrl-2> for the ^{2} symbol and <Ctrl-Shift-2> for the $\sqrt{}$ symbol. Don't worry that you can't see the whole equation as you type it in. It is still there. After you have carefully entered the equation, move over to the Constants section by hitting the <F6> key. First we need to indicate which variables are to be on the horizontal and vertical axes. For "x=" type the letter "h" (for horizontal), then arrow down to "y=" and type the letter "v" (for vertical). Now let's pick some values for the other constants. For now let's use x1=-4, y1=0, x2=4, y2=0, c=10.

Interpreting the graph
We are now ready to graph the ellipse. When you hit <Ctrl-Enter> to graph

it, you will notice that the powerful graphing ability comes at the cost of speed. The way the Relation Plotter works is by checking each point in the plane to determine how the result compares to zero. If it is greater than zero, the screen is shaded solid grey. If it is less than or equal to zero, it is shaded with the dot pattern. If the result is not real, no shading takes place at all. Therefore, the graph itself requires some interpretation. The shape of the relation in question should then be the border between the two shaded patterns as is easily seen with the ellipse. This slow process can be speeded up considerably without severe loss in accuracy by increasing the Horizontal and Vertical Sample Size values. This value tells the plotter how many pixels to skip before sampling the equation.

Now that we understand what the graph means, we are now ready for the case where the distances are multiplied. Press the <F2> key to get back to the equation. Rather than typing it in all over again, just arrow over to the "+" sign in the middle of the formula and change it to an "*" sign so that it is now "$\sqrt{((x-x1)^2+(y-y1)^2)} * \sqrt{((x-x2)^2+(y-y2)^2)}$ - c" and graph it. You may find the results surprising. Now go back and make c=15 and graph it again. Try c=17, then c=16, and finally c=50. (I purposely did not put the resulting graphs here to maintain at least some element of surprise.)

Finally, go back and change the equation again to reflect the second student's question about the shape where the three distances add to a constant. The equation is a long one so be careful typing it in:
"$\sqrt{((x-x1)^2+(y-y1)^2)} + \sqrt{((x-x2)^2+(y-y2)^2)} + \sqrt{((x-x3)^2+(y-y3)^2)}$ - c"
Try various combinations. Make the three points collinear, equilateral, isosceles, random, etc... Make "c" small, then big.

The possibilities are endless. You can now determine the shape of graphs that cannot be solved for explicitly. For example, what does x^y=y^x look like? Use 0=x^y-y^x and find out. We had a class contest on who could come up with the "coolest" looking relation. My entry was y=|x|^sin(x-y). (Try solving that one for x or y!)

<u>System of Inequalities</u>
An unexpected consequence came out of the Relation Plotter. Since the program evaluates comparisons such as "x>5" as either being 1 (true) or 0 (false), it is possible to directly enter a system of inequalities. For example: "y<3x and y>x^3 and x>0 and |x*y|<1" generates a solid shaded region where the statement is true.

Bifurcations

One of the earliest manifestations of the science of Chaos was with bifurcations dealing with population growth. In Algebra II or in Pre-Calculus, oversimplified recursive models of population growth are often studied. Typically, the model states that the following year's population is proportional to the current year's population. That is: $p_{next}=r*p$ where p is the population and r is the proportionality constant. This model leads to either exponential growth or decay depending on whether r is greater than or less than 1. This population model is a good place to start, but doesn't take into consideration factors such as overcrowding, food supply, etc... A better model is given by the recursive formula $p_{next}=r*p*(1-p)$ where p is the percentage of the maximum population that can exist and r is some constant. Notice that if the population becomes very large, then the next year's population will be small since 1-p will be small. A few iterations on a hand calculator will show that if $0 \le r \le 1$, then the population will eventually become extinct, regardless of the initial population (lim n->∞ p_n=0). When $1 < r \le 3$, the population eventually approaches a constant value (lim n->∞ p_n=k). For example, when r=2.5, p->0.6 after many years, regardless of the initial population. When r>4, the model breaks down, sending p toward negative infinity.

A really interesting situation occurs when 3<r<4. As r gets bigger than 3, p does not converge to a single value but eventually starts alternating between two values. For example, when r=3.3, then after many years have gone by, p alternates between about 0.478 and 0.823 every other year. As r continues to increase, after many years p starts alternating between four different values every four years, then eight, then sixteen, and so on. This splitting phenomenon is where the name "bifurcations" is derived. It has been found that the values of r when the split occurs form a converging geometric series. When r reaches this limit, the value of p after many years completely loses any pattern what-so-ever and is said to be "chaotic." However, at about r=3.8285, p suddenly starts repeating every third year. As r continues to increase, the bifurcations start occurring again, splitting from 3 into 6, then 12, 24, and so on. Even though it has no real significance to a population model, the same sort of chaotic behavior occurs when r is negative.

Entering a Bifurcation formula

To use the formula discussed above, enter "r*p*(1-p)" and make sure that in the Constants section that r is the independent variable (enter the letter "i") and p is the dependent variable (enter the letter "d"). Since bifurcations graphs seek to determine what happens after a large number of iterations, you want it to calculate a lot of iterations before plotting anything. This is done by setting the pre-plot loops to a value like 100 or more. Then set the number of iterations to graph to a somewhat smaller value, say about 50 or

so. Finally, it is important to give some initial value. In the case of the population formula, it doesn't matter what the value is as long as it is not zero. Try 0.5 for a safe value. To plot it, press <Ctrl-Enter> as before, but be prepared for a long wait. The graphing process requires many calculations so be patient.

Another good formula to try is the Mandelbrot equation: $z_{next}=z^2+c$ with c as the independent variable and z as the dependent with an initial value of zero. Many other equations demonstrate bifurcation behavior. Any function with a maximum or minimum point can have this phenomenon. For example $x_{next}=r*\sin(\pi*x)$ works well with a non-zero initial value.

Recursive Relations

This module is the newest in the WL-Plot program. It got its start when a student brought an article to me about a newly discovered type of fractal. As I studied this new finding, I found that it would be very beneficial to have such a plotting module in WL-Plot.

A recursive relation is similar to a regular recursive function, except that both the x coordinate and the y coordinate each have their own formulas and can depend on each other. Often, the relation will form some sort of pattern. If the iteration of (x,y) points seem to get pulled into a pattern, an "attractor" is said to exist. Some attractors form points or circles while others can be fractals. If the attractor is a fractal, it is referred to as a "strange attractor."

For example, a well known one is called the Henon Attractor defined by: $x_{next}=1+y-a*x$ and $y_{next}=b*x$, where x is the horizontal variable (use the letter "h"), y is the vertical variable (use the letter "v"), a=1.4, and b=0.3. Set pre-plot loops=100 and plot loops=10000. Graph this relation and then zoom in to the region where the points are being plotted and you will see a horse-shoe shaped relation. If you continue zooming in to part of the relation, you'll see that the relation is split into multiple lines as show in the second graph.

Another really nice relation is defined by $x_{next}=1-y+|x|$ and $y_{next}=x$. When graphed with an initial x value=-0.1 and initial y value=0, it produces the relation on the right. It is referred to the Gingerbread Man Attractor for obvious reasons. The interesting thing about this relation is that many initial values produce this same shape.

Curve Fitting

The Curve Fitting module of WL-Plot is useful for determining what equation would best match a set of data points. It was first written for my physics students to use it to help analyze their data from labs. Simply enter the number data points that you have and then enter the (x,y) pairs of data points. After all the points have been entered, press the <F3> key to move to the Plot Type section. Select the mathematical model which you think best matches the data. Alternatively, you can select Best Fit Curve to let the WL-Plot try out all the models and select the best one. When you select the Polynomial model, you must also indicate what degree polynomial to use. In addition to calculating the constants for the best fit equation, WL-Plot also calculates the correlation squared term, R^2, which varies from 0 to 1, with 1 being a perfect fit.

Let's start with the Number of Data Points set equal to 5. After you type in the number of data points and hit <Enter>, you will be put into the Edit Data Points mode. Try entering the following data points: (1,2), (3,4), (5,7.5), (6.5,8), (9.5,9.5). This is done by typing "1" then press the <Enter> key, "2" <Enter>, "3" <Enter>, etc... You can go back and edit a number if necessary by pressing the <Up> or <Down> arrow keys to move up and down. Press the <Tab> key to move back and forth between the X and Y values. The can be sorted according to the X or Y values by pressing the <Alt-X> or <Alt-Y> keys respectively while in Edit Data mode. The data can be sorted in reverse order by selecting <Alt-Shift-X> or <Alt-Shift-Y>.

Once you have pressed <Enter> after the last data point, the best fit line is calculated. If you used the points above, the best fit line equation y=Ax+B occurs when A=0.911765 and B=1.64118 with an R^2=0.922477 as is shown on the right side of the screen.

When you select the Plot option (or use the short-cut <Ctrl-Enter> key), another menu is presented in which the user can enter titles, grid intervals, axis titles, etc... Many of the values are determined automatically but can be overridden by turning off the "Automatic X-Y Values" option. When everything is as desired, select the Plot graph option and the data points and best fit curve are shown. You cannot zoom in or out with the Curve Fitting module.

To get back to the first menu, press the <Esc> key. To see what the best fit parabola would be, move to the "Plot Type" section by pressing the <F3> key, arrow down to the "Polynomial" option, and press <Enter>. You will then be prompted for the degree of the polynomial. Since we want a parabola, type "2" and press <Enter>. You will notice that the formula is

now $y=A_0+A_1x+A_2x^2$ where $A_0=0.0775381$, $A_1=1.77165$, $A_2=-0.0816647$, with $R^2=0.977147$. The higher R^2 value means that it is a better fit.

Now we'll try and even higher degree polynomial. Since there are only 5 data points, it will perfectly describe a degree 4 polynomial. Changing the degree to 4 gives $A_0=5.1926$, $A_1=-5.62613$, $A_2=2.84086$, $A_3=-0.427992$, $A_4=0.0206651$, with $R^2=1$ (a perfect fit).

The other mathematical models available are exponential, logarithmic, and power. If you are using the exponential model, make sure all of the Y values are positive.   With the logarithmic model, all of the X values must be positive. With the power model, both X and Y values must all be positive, ie: don't use (0,0). Selecting the "Best Fit Curve" option causes the program to try out all the models and use the equation which produces the highest R^2 value.

Two other mathematical options are available, None, and Connect Dots. The None option only plots the points and nothing else. Connect Dots will connect the data points in the order listed. This is the only option whose outcome depends on the order of the data.   To sort the data, select the Edit Data option and then press the <Alt-X> or <Alt-Y> to sort on the X or Y data.

Plotting Multiple Graphs on One Screen:
In order to compare data, it is often useful to be able to show more than one set of data at a time. With WL-Plot, this is possible, but takes a little prior planning. Using the same data you entered above, select the Plot option to move to the Plot Menu.   MAKE SURE YOU TURN THE Automatic Values OPTION OFF. If necessary, increase the maximum X and Y values to include whatever (x,y) data values that you wish to plot on top of the data you already have. Plot the graph as before and then return to the Curve Fit Menu. Now make whatever changes in data or mathematical model you desire and select Plot to get to the Plot Menu. Now move all the way to the bottom and turn the Overlay Graph on. IT IS IMPORTANT NOT TO CHANGE ANY OF THE TITLES OR NUMERIC VALUES. The only other thing that you can change is the Plot Character to distinguish from the different sets of data. There is no limit to how many sets of data you can stack on top of each other.

Printing Graphs

To print a graph directly to a printer, DOS's GRAPHICS utility must be loaded before starting WL-Plot. This is done automatically for you if the SETUP program has been run. Note that DOS version 3.3 and earlier can only print CGA mode graphics while version 4.0 and later can print EGA and VGA graphics as well. You can force the plotting program to use CGA mode graphics by starting the program with "wlplot cga". This allows you to print the screen if your DOS version is prior to 4.0 even if you have a VGA card. You can also print a screen by saving it to a file by selecting "Save-PCX" and then importing the PCX file into a painting or word processing program. This allows you to merge plots with other graphics into a document. (See "Saving Your Work" above.) WL-Plot is also compatible with the Microsoft Windows (tm) "copy to clipboard" feature using the <PrntScrn> key.

More Information

Hopefully this tutorial has gotten you started in the right direction. For more information about WL-Plot, see the included files WLPLOT.DOC and WLPLOT.HLP.

Appendix A - Special Characters:

To produce the nonstandard special characters, use the indicated key combinations.

Name	Symbol	Key Stroke	
alpha	α	<Ctrl-A>	
beta	ß	<Ctrl-B>	
delta	δ	<Ctrl-D>	
epsilon	ε	<Ctrl-E>	
phi	φ	<Ctrl-F>	
Phi	Φ	<Ctrl-Shift-F>	
gamma	τ	<Ctrl-G>	
Gamma	Γ	<Ctrl-Shift-G>	(gamma function)

infinity	∞	<Ctrl-I>	(big value)
mu	μ	<Ctrl-M>	
omega	Ω	<Ctrl-O>	
pi	π	<Ctrl-P>	(3.14159…)
sigma	σ	<Ctrl-S>	
Sigma	Σ	<Ctrl-Shift-S>	
theta	Θ	<Ctrl-T>	
squared	²	<Ctrl-2>	(square function)
sq-root	√	<Ctrl-Shift-2>	(square-root function)

Appendix B - Built in Functions:
The following functions and constants are predefined by WL-Plot.

Constants:
    3.141592653589793 pi or π <Ctrl-P>
    2.718281828459045 e
    3.402823466e+38   inf or infinity or ∞ <Ctrl-I>
(used when "really big" number is needed)

Variables:
use variable   p   (use variable p)
save in variable   ->p (save in variable p)

Binary functions:
    add   x+y
    subtract   x-y
    multiply   x*y or 2x
    divide   x/y
    exponentiation x^y
    modulus   x mod y or mod(x,y)
    combination   comb(n,r)
    permutation   perm(n,r)
    maximum   max(x,y)
    minimum   min(x,y)
    random   rand(x,y)

Unary functions:
    negation   -x or neg(x)
    inverse   inv(x)
    square root   sqrt(x) or √x
    square   sq(x) or x²
    round   round(x)
    ceiling   ceil(x)   round up
    floor   floor(x)   round down
    integer   int(x) truncate decimal
    fractional   frac(x)
    absolute value abs(x)
    sign   sign(x)
    factorial   fact(n) or n!
    gamma   gamma(x) or Γx
    is value real?   isreal(x)

Trigonometry functions:

sine	sin(x)
inverse sine	asin(x) or arcsin(x)
cosine	cos(x)
inverse cosine	acos(x) or arccos(x)
tangent	tan(x)
inverse tan	atan(x) or arctan(x)
inverse tan2	atan2(x,y) or arctan2(x,y)
cotangent	cot(x)
inverse cot	acot(x) or arccot(x)
secant	sec(x)
inverse sec	asec(x) or arcsec(x)
cosecant	csc(x)
inverse csc	acsc(x) or arccsc(x)
rad to deg	raddeg(x)
deg to rad	degrad(x)

Logarithm related functions:

natural log	ln(x)
exponential	exp(x)
common log	log(x)
antilog	alog(x) or antilog(x)
log base n	logb(n,x)
hyperbolic sin	sinh(x)
inverse sinh	asinh(x)
hyperbolic cos	cosh(x)
inverse cosh	acosh(x)
hyperbolic tan	tanh(x)
inverse tanh	atanh(x)

Bessel functions:

1st kind, 0	j_0(x)
1st kind, 1	j_1(x)
1st kind, n	j_n(n,x)
2nd kind, 0	y_0(x)
2nd kind, 1	y_1(x)
2nd kind, n	y_n(n,x)

Trinary Function:

conditional if	if(b,t,f)

If b is true (positive) then t will be use. If b is false (not-positive) use f.
     Ex: if(x>=0,3x,x^2)
The "if" is probably the single most powerful function here. It allows the user

to perform branching of sorts.   Nested if's are allowed.   For example the following pseudo-code

        if x < 2
                then y = 3x^2
        else if x < 5
                then y = 6x
        else
                y = x^2+5

can be coded Algebraically as
        if(x<2, 3x^2, if(x<5, 6x, x^2+5))
or in RPN as
        x 2 < 3 x sq * x 5 < 6 x * x sq 5 + if if

Boolean Comparisons: (evaluates to 1 if true, 0 if false)
    less than   a<b
    greater than   a>b
    less than or equal to a<=b
    greater than or equal to   a>=b
    equal to   a=b
    not equal to   a<>b
    and   a and b   or   and(a,b)
    or   a or b   or   or(a,b)
    exclusive or   a xor b   or   xor(a,b)

The result of a boolean comparison evaluates to 1 if true, 0 if the result is false. In numeric comparisons, non-real values cannot be compared except in the case of "<>" when a real and a non-real are always considered to be not equal.   Otherwise, reals compared to non-real values are considered false. This leads to the interesting result that "sqrt(-1)>0" and "sqrt(-1)<0" are both considered false, and "not(sqrt(-1)>0)" and "not(sqrt(-1)<0" are both considered true.

Stack Functions: (available in RPN mode only)
    duplicate bottom element   dup   or push
    drop bottom element   drop or pop
    swap bottom two elements swap
    recall stack element n   n rcl
    store in stack element n   n sto
    roll bottom n elements   n roll
     n must be a number.
     If n is positive the stack rolls down, a negative n rolls up

Appendix C - Order of Operations:

Algebraic Order of Operations:
highest   parentheses ()
        functions
        ^
        implied multiplication
        implied parentheses following functions
        * / mod
        + -
        = <>
        < <= > >=
        and
        xor
        or
lowest   ->